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Nonlinear critical layers and their development in streaming-flow stability

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The effect of increasing disturbance size on the stability of a laminar streaming flow is considered theoretically at high Reynolds numbers Re. The theory has a rational basis that allows detailed understanding of the delicate physical balances controlling stability, and is presented with an accelerating boundary layer taken as the basic flow. The theory predicts that the scales and properties required to produce the Rayleigh situation (where the disturbances have wave speed and wavelength comparable to the typical speed and thickness respectively of the basic flow) in neutral stability are very different from those predicted by a classical approach, involving a relative disturbance size $O(Re^{-\frac{1}{6}})$ rather than the classical suggestion $O(Re^{-\frac{1}{3}})$. Before then, however, the disturbances undergo an abrupt alteration in scale and character as they pass through the just slightly smaller size $O(Re^{-\frac{7}{36}})$, with the stability structure changing from the relatively large-scale form of linear theory to the more condensed Rayleigh form by means of a nonlinear interaction within the critical layer. Strong higher harmonics of the fundamental disturbance are induced throughout the flow field by the velocity jump across the critical layer, but the phase jump remains the most significant property. Solutions for the nonlinear critical layer are recalculated and reanalysed. Also, the mean-flow correction produced by the nonlinear critical layer is shown to be smaller than the main part of the fundamental, owing to the regularity of the latter. As the Rayleigh stage is approached, the lateral variation of the induced pressure force through the critical layer begins to exert a considerable influence. Similar characteristics also arise in other fundamental streaming flows, and the implied Rayleigh stage is the subject of a subsequent investigation.

1. Introduction

Our intention is to further the self-consistent structured account of the nonlinear stability of boundary-layer flows in the limit of high Reynolds numbers, the only limit where the basic flow is truly of a boundary-layer form. This work is therefore in the same spirit as Smith's (1979*b*) study of the stability properties near the lower branch of the neutral curve at large Reynolds numbers. There, as the size of the disturbance is increased, the first fully nonlinear stage reached is governed by the three-zoned structure and scalings of the triple-deck, if the basic flow is a boundary layer. The same structure also describes the linear parallel or non-parallel flow theory of Tollmien–Schlichting waves in fact, along the lower neutral branch (Smith 1979*a*; Hall & Smith 1982; cf. Bouthier 1973; Gaster 1974).

Here we consider the effects of increasing the size of the disturbance, relative to the inverse powers of the large Reynolds number, but along the upper branch of the neutral stability curve for a general accelerating planar boundary layer on a fixed wall. The main differences from the lower branch case arise from the well-known distinction of the critical layer and the viscous wall layer in the balance of forces controlling stability along the upper branch. The structure of this balance is set out briefly in § 2, for the linear theory of an infinitesimal disturbance first. It is a five-zoned structure similar to that shown by Bodonyi & Smith (1981) to govern the upper branch parallel and non-parallel flow stability of the non-accelerating Blasius boundary layer. It provides the foundation for the subsequent study of nonlinear aspects necessarily induced as the disturbance size increases. For the linear structure continues to dictate the neutral stability properties until the relative disturbance size, δ say, rises to $O(Re^{-\frac{7}{36}})$, where Re denotes the global Reynolds number (see (1.1) below). Even at that stage, $\delta = O(Re^{-\frac{7}{36}})$, the stability structure remains substantially unaltered (see § 3) *except that* the critical layer close to the wall becomes influenced strongly by nonlinear interactions, higher harmonics of the fundamental disturbance appear significantly everywhere in the flow field and the wall layer becomes nearly nonlinear. The new critical-layer balance then corresponds to the proposals of Benney & Bergeron (1969), Davis (1969), Haberman (1972) and others, but subject to the modified boundary conditions revealed by Brown & Stewartson (1978). Despite the production of the higher harmonics everywhere, the concept of a phase shift across the nonlinear critical layer remains important, since the phase shift continues to control the neutral stability state in the nonlinear regime, and calculations of the nonlinear critical-layer problem are consistent with Haberman's (1972) numerical results. As the disturbance size increases further within the $O(Re^{-\frac{7}{36}})$ stage, however, the neutral wavelength falls sharply and the frequency and wave speed rise as inviscid effects begin to dominate the critical-layer balance and a cat's eye of uniform vorticity starts to form. The modifications required of Haberman's (1972) and Brown & Stewartson's (1978) analyses of this process are described in § 4. The present work puts the nonlinear critical-layer effect into context, explicitly determines the amplitude dependence of the neutral waves, the velocity jump across the critical layer, and the higher harmonics induced; it shows that the mean-flow correction is *smaller* than the fundamental disturbance, contrary to earlier suggestions, and it reaffirms the importance of the phase shift. It also gives a simpler method (§ 4) for determining the velocity jump in the inviscid limit. However, the main inference to be drawn is that presented in § 5.

There it is anticipated that a great change must take place in the stability structure when the disturbance size is increased just slightly, from $O(Re^{-\frac{7}{36}})$ to $O(Re^{-\frac{1}{6}})$. For then the critical layer moves out into the midst of the basic flow and the neutral wavelength becomes comparable with the characteristic lateral dimension of the basic flow. Therefore the classical inviscid Rayleigh situation is recovered, apparently. Yet just as that happens the nonlinear critical layer develops another new feature, involving the setting up of a significant lateral pressure gradient across the critical layer. In addition, the wall layer then becomes nonlinear. So it may be concluded that an entirely different balance of forces is provoked in the Rayleigh situation encountered as the stage

$\delta = O(Re^{-\frac{1}{6}})$ is reached. There is little doubt that the same conclusion holds also in other streaming flows such as the Blasius flow and plane Poiseuille flow, as the Rayleigh situation is reached with δ of order $Re^{-\frac{1}{6}}$, and there may be applications elsewhere as well, in stratified fluid flows for example (Kelly & Maslowe 1970; Maslowe 1972, 1977; Stewartson 1978; Brown & Stewartson 1978, 1980*a*). It seems worth remarking that the classical approach would tend to suggest that the Rayleigh situation is encountered when the disturbance size is only $O(R^{-\frac{2}{3}})$, where $R = Re^{\frac{1}{2}}$ is a local Reynolds number, in contrast with the above prediction of $O(R^{-\frac{1}{3}})$. The anticipated Rayleigh situation will be studied by Smith & Bodonyi (1982), hereinafter referred to as part 2.

All the stability properties of importance below are sufficiently local that the wall may be taken to be flat, lying along the x-axis, say, where x, y are Cartesian co-ordinates. The corresponding components of the total velocity field are written $\bar{u}$, $\bar{v}$ and the pressure is $\bar{p}$. Here x, y, $\bar{u}$, $\bar{v}$, $\bar{p}$ and the time t are non-dimensionalized with respect to l, l, u_∞, u_∞, ρu_∞^2 and lu_∞^{-1} in turn, where ρ is the fluid density and l, u_∞ respectively are characteristic streamwise length and velocity scales of the basic flow. To be specific we may take the constant u_∞ to be the local free-stream speed just outside the boundary layer that is set up when the Reynolds number Re defined by

$$Re = u_\infty l/\nu \tag{1.1}$$

is large. The fluid is supposed to be incompressible with kinematic viscosity ν, and its unsteady motion is assumed to be two-dimensional. Extensions of the theory can be made, in principle at least, to include three-dimensionality for instance, to treat other basic flows which (like the accelerating boundary layer) are linearly destabilized only by the presence of viscosity, or to examine slow growth or decay of the disturbance amplitude instead of the neutral criteria to be established here. The establishment of the neutral criteria for the present case, and their dependence on disturbance size, is, however, regarded as a useful first step for the systematic theory. On the other hand the influence of initial conditions as pointed out by Stewartson (1978) seems to be a very important factor in the development of nonlinear critical layers and needs further consideration, despite the apparent agreement obtainable between stability theories which take no account of initial values and experiments or calculations aimed at following the flow response for a maintained disturbance of fixed frequency, say (see for example Gaster 1974; Murdock 1977; Fasel, Bestek & Schefenacker 1977; Smith 1979*a*, *b*).

2. Linear-disturbance structure

The springboard for the present paper and its sequel (part 2) is the solution structure, near the upper branch of the neutral stability curve when the Reynolds number Re is asymptotically large, for an infinitesimal disturbance to the basic streaming flow. Although the main properties of such a disturbance have been known for many years from classical linear-stability theory (Lin 1955, Stuart 1963, Reid 1965) their determination has largely been carried out in a rather *ad hoc* fashion. By contrast a systematic treatment is much more illuminating of the delicate physical and mathematical balances controlling instability. The systematically based structure near the upper branch has not been fully recognized in the past, but has been set out recently by Bodonyi & Smith (1981) and Smith & Bodonyi (1980) for the Blasius boundary layer and entry flow in a channel or pipe. The present study concentrates on the stability of general accelerating boundary layers. This is done to complement partly the earlier work just referred to by presenting the more general case, for a linear disturbance, but more especially to make use of the less involved balances governing the non-Blasius accelerating flow stability when we come to nonlinear aspects later.

The substantial differences between the stability features of the accelerating and the Blasius boundary-layer flows stem from the presence of a favourable pressure gradient in the former flow. As a consequence the curvature of the basic velocity profile is now non-zero at the wall, whereas for the Blasius flow (Bodonyi & Smith 1981) it is zero, and, since the curvature at the critical layer near the wall (see below) controls the stability properties, the associated scalings are now quite distinct from those holding in the Bodonyi & Smith (1981) study. The orders of magnitude which provide the clues to the scalings and disturbance structure could all be derived from the classical linear theory, however, just as in the Blasius case. The disturbance structure consists of five zones I–V governing the lateral variation of the solution, has streamwise length scale $O(Re^{-\frac{5}{12}})$, and is depicted in figure 1. Zones I–V have thicknesses of the orders $Re^{-\frac{1}{2}}$, $Re^{-\frac{7}{12}}$, $Re^{-\frac{23}{36}}$, $Re^{-\frac{2}{3}}$ and $Re^{-\frac{5}{12}}$ respectively. So the lateral and streamwise extents of the structure are much greater than the boundary-layer thickness. Again, the physical characteristics of the five zones are analogous to those described for Blasius flow stability by Smith & Bodonyi (1980) and Bodonyi & Smith (1981).

The equations to be addressed, in this section only, are the linearized Navier–Stokes equations

$$\operatorname{div}\mathbf{q} = 0, \quad \partial\mathbf{q}/\partial t + (\mathbf{Q}.\nabla)\,\mathbf{q} + (\mathbf{q}.\nabla)\,\mathbf{Q} = -\nabla p + Re^{-1}\nabla^2\mathbf{q} \tag{2.1}$$

for the infinitesimal disturbance $\delta\mathbf{q} = \delta(u, v), \delta p$ obtained by expanding the total velocity and pressure field in the form

$$[\bar{u}, \bar{v}, \bar{p}]\,(x, y, t) = [U, V, P]\,(x, y) + \delta[u, v, p]\,(x, y, t) + O(\delta^2), \tag{2.2}$$

and working to order $\delta \ll 1$ in the Navier–Stokes equations. The boundary conditions require no slip at the wall and boundedness of the disturbance in the far field. Since the steady basic flow $\mathbf{Q} = (U, V), P$ has a boundary-layer character only when $Re \to \infty$ – a fact traditionally ignored in the 'parallel-flow approximation' of course (see Smith 1979*a*, *b*) – a severe restriction is strictly necessary on the present disturbance size δ, namely

$$\delta \ll Re^{-N} \quad \text{for all} \quad N > 0, \tag{2.3}$$

in order that (2.1) may remain valid when the basic flow has its boundary-layer form, as in Smith (1979*a*, *b*). Lifting the restriction (2.3), to allow for bigger disturbances, leads to the nonlinear aspects examined in §§ 3–5 below and in part 2.

For $Re \gg 1$, then, the basic flow takes the boundary-layer form

$$U(x, Y) = U_B(x, Y) + \ldots, \quad V(x, y) = O(Re^{-\frac{1}{2}}), \tag{2.4a}$$

say, where $Y = Re^{\frac{1}{2}}y$ is the boundary-layer co-ordinate. The accelerating boundary-layer profile $U_B(x, Y)$ has the properties

$$U_B(x, \infty) = 1, \quad U_B(x, Y) \sim \lambda(x)\,Y + \lambda_2(x)\,Y^2 + \ldots \quad \text{as} \quad Y \to 0, \tag{2.4b}$$

where $2\lambda_2(x) = \partial P/\partial x(x, 0) < 0$ relates the local favourable pressure gradient to the profile curvature $\lambda_2(x)$, and $\lambda(x) > 0$ gives the local skin friction. Since the variation

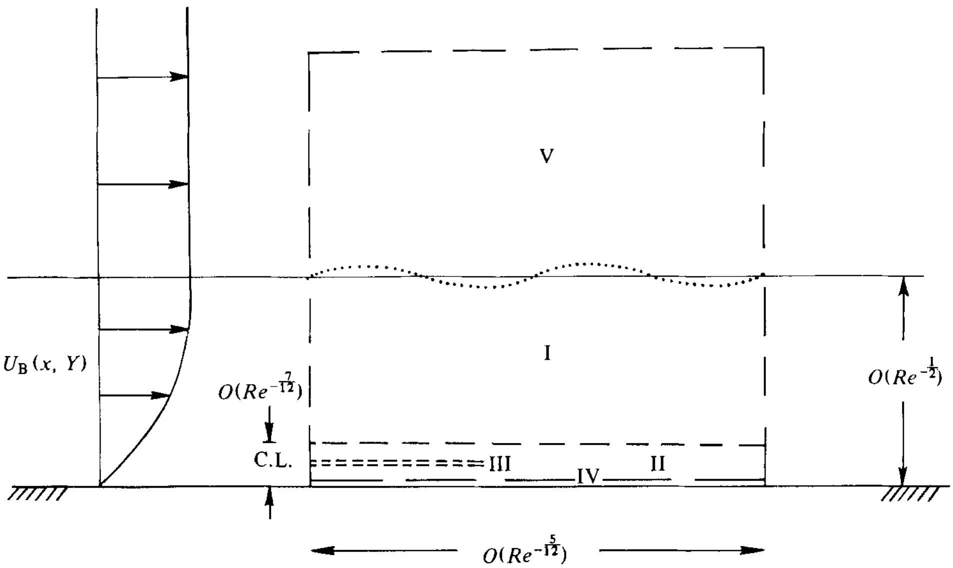

FIGURE 1. Sketch (not to scale) of the relatively large five-zoned structure of a travelling wave disturbance of amplitude $\delta \leqslant O(Re^{-\frac{7}{36}})$ in an accelerating boundary layer (see §§ 2, 3), with basic velocity profile $U_B(x, Y)$. Here C.L. denotes the critical layer III.

of the disturbance in (2.1) occurs on the relatively short streamwise length scale of $O(Re^{-\frac{5}{12}})$ the variation of U_B with x plays only a passive role, and x can be regarded as a parameter provided that, as below, only the first few leading terms of the disturbance solution are required (Bouthier 1972, 1973; Gaster 1974; Smith 1979*a*; Bodonyi & Smith 1981).

Returning to the disturbance properties, for convenience we set $\epsilon = Re^{-\frac{1}{12}}$. Then the streamwise and temporal variations of the disturbance depend mainly on X, τ, where $x = \epsilon^5 X$, $t = \epsilon^4\tau$[see Reid 1965, equations (3.121), (3.123)], although a multiple-scales replacement of $\partial/\partial x$ by $\epsilon^{-5}\,\partial/\partial X + \partial/\partial x$ would be called for to treat the non-parallelism of the basic flow (Smith 1979*a*; Bodonyi & Smith 1981). In the zones I, II, IV, V the disturbance develops in the forms

$$(u,v,p) = \left\{\begin{array}{l} E[u_0, \epsilon v_0, \epsilon p_0] + \ldots + \text{c.c.} \quad \text{in I,} \quad \text{where} \quad y = \epsilon^6 Y, \\ E[u^{(0)} + \epsilon u^{(1)}, \epsilon^2 v^{(0)} + \epsilon^3 v^{(1)}, \epsilon p^{(0)} + \epsilon^2 p^{(1)}] + \ldots + \text{c.c.} \quad \text{in II, where} \\ \hfill y = \epsilon^7 \tilde{Y}, \\ E[\tilde{u}_0, \epsilon^3 \tilde{v}_0, \epsilon P_0] + \ldots + \text{c.c.} \quad \text{in IV,} \quad \text{where} \quad y = \epsilon^8 Z, \\ E[\epsilon \hat{u}_0, \epsilon \hat{v}_0, \epsilon \hat{p}_0] + \ldots + \text{c.c.} \quad \text{in V,} \quad \text{where} \quad y = \epsilon^5 \hat{y}, \end{array}\right\} \qquad (2.5a)$$

where c.c. denotes the appropriate complex conjugate and

$$E \equiv \exp\left(i(\tilde{\alpha}X - \tilde{\beta}\tau)\right).$$

The reduced wavenumber $\tilde{\alpha}$ and frequency $\tilde{\beta}$ are also to be expanded,

$$\tilde{\alpha} = \alpha_0 + \epsilon\alpha_1 + \ldots, \quad \tilde{\beta} = \beta_0 + \epsilon\beta_1 + \ldots, \qquad (2.5b)$$

and for definiteness $\tilde{\alpha}, \tilde{\beta}$ will be taken to be real so that our concern is with the neutrally stable situation or travelling-wave disturbance. An extension of the analysis to yield growth rates instead can be readily made. The basic flow is given by (2.4a) in I and so has the forms

$$(U, V) = \begin{cases} [\epsilon\lambda\tilde{Y} + \epsilon^2\lambda_2\tilde{Y}^2 + \ldots, O(\epsilon^8)] & \text{in II,} \\ [\epsilon^2\lambda Z + O(\epsilon^4), O(\epsilon^{10})] & \text{in IV,} \\ [1, 0] + O(\epsilon^6) & \text{in V.} \end{cases} \tag{2.5c}$$

Substitution of (2.5a–c) into (2.1) produces the successive disturbance solutions

$$u_0 = A_0 U_{BY}, \quad v_0 = -i\alpha_0 A_0 U_B, \quad p_0 = p^{(0)} = P_0; \tag{2.6a}$$

$$\left.\begin{aligned} u^{(0)} &= \lambda A_0, \quad v^{(0)} = -i\alpha_0 A_0 \lambda(\xi + \tilde{c}); \quad \text{for} \quad \xi > 0, \\ v^{(1)} &= i\hat{D}^{(1)} - i\alpha_0 \bar{A}_1 \lambda(\xi + \tilde{c}) - i\alpha_0\lambda_2 A_0 \xi(\xi + 2\tilde{c}\ln\xi); \end{aligned}\right\} \tag{2.6b}$$

$$\tilde{u}_0 = \alpha_0 P_0 \beta_0^{-1}[1 - \exp(-mZ)], \quad \tilde{v}_0 = i\alpha_0^2 P_0 \beta_0^{-1}[m^{-1} - Z - m^{-1}\exp(-mZ)]; \tag{2.6c}$$

$$\hat{p}_0 = P_0 \exp(-\alpha_0 \hat{y}), \quad \hat{v}_0 = -iP_0 \exp(-\alpha_0 \hat{y}) \quad (m \equiv \beta_0^{\frac{1}{2}} e^{-\frac{1}{4}i\pi}) \tag{2.6d}$$

Here A_0, P_0 are constants representing the disturbance displacement and pressure amplitudes, $\tilde{c}\lambda = \beta_0/\alpha_0 = c_0$ is the scaled wave speed, $\xi \equiv \tilde{Y} - \tilde{c}$, and the constants A_0, $\hat{D}^{(1)}$, $\bar{A}_1$ can be taken to be real without loss of generality. The appearance of the final logarithmic term in (2.6b), owing to the profile curvature effect $\lambda_2(x)$ of (2.4b), is vital since it forces the smoothing out in the viscous critical layer III surrounding $\xi = 0$, $\tilde{Y} = \tilde{Y}_c = \tilde{c}$. Classical linear theory (Lin 1955; Stuart 1963) shows that a phase shift of $-\pi$ results there (cf. § 3), so that (2.6b) holds below the critical layer provided we replace $\ln\xi$ (for $\xi > 0$) by $\ln|\xi| - i\pi$ (for $\xi < 0$). For later reference we note that, in III, $y = \epsilon^7\tilde{Y}_c + \epsilon^{\frac{23}{3}}Y_1$, with Y_1 of $O(1)$, i.e. $\xi = \epsilon^{\frac{2}{3}}Y_1$, and

$$v = E[\{\epsilon^2 v^{(0)}|_{\tilde{Y}=\tilde{Y}_c} + \epsilon^{\frac{8}{3}} v^{(0)}_{\tilde{Y}}|_{\tilde{Y}=\tilde{Y}_c} Y_1 + \epsilon^3 v^{(1)}|_{\tilde{Y}=\tilde{Y}_c} + \epsilon^{\frac{11}{3}}\tilde{V}\} + \ldots] + \text{c.c.} \tag{2.7}$$

from (2.5a), (2.6a, b). Here the function $V(Y_1)$ exhibits logarithmic behaviour only as $Y_1 \to \pm\infty$, and the solution of the Airy equation governing $\partial^2\tilde{V}/\partial Y_1^2$ leads to the phase shift of $-\pi$ (cf. §§ 3–5). Matching u, v, p between zones II, IV and between I, V, together with the momentum balance in zone II, then requires the relations

$$v^{(1)}|_{\tilde{Y}=0+} = \frac{i\alpha_0^2 P_0}{m\beta_0}, \quad P_0 = \alpha_0 A_0, \quad \lambda\beta_0 A_0 = \alpha_0 P_0, \tag{2.8a,b,c}$$

in turn. So the real parts of (2.8a) give

$$2\lambda_2\alpha_0 A_0 \tilde{c}^2\pi = -\alpha_0^2 P_0 (2\beta_0^3)^{-\frac{1}{2}}; \tag{2.9}$$

the imaginary parts only fix a first relation between the next-order terms α_1, β_1 of (2.5b). Therefore (2.8b, c), (2.9) fix α_0, β_0 and c_0, and we obtain the expressions

$$\alpha_0 = \frac{\lambda^{\frac{11}{6}}}{2^{\frac{1}{2}}(-\pi\lambda_2)^{\frac{1}{3}}}, \quad \beta_0 = \frac{\lambda^{\frac{8}{3}}}{2(-\pi\lambda_2)^{\frac{2}{3}}}, \quad c_0 = \frac{\lambda^{\frac{5}{6}}}{2^{\frac{1}{2}}(-\pi\lambda_2)^{\frac{1}{3}}} \tag{2.10a}$$

for the reduced wavenumber, frequency and wave speed of a neutrally stable disturbance. The corresponding unscaled quantities (i.e. in terms of x, t rather than X, τ) are

$$\alpha = \epsilon^{-5}\alpha_0, \quad \beta = \epsilon^{-4}\beta_0, \quad c = \epsilon c_0, \tag{2.10b}$$

to leading order.

The criterion (2.10a) agrees with the classical linear results (e.g. in Reid 1965; Drazin & Reid 1980), and it verifies the importance of the profile-curvature effect λ_2

through its controlling of the phase jump $-\pi$. Again, because of the derivation of (2.10*a*) the necessity of the pressure–displacement balancing throughout the whole disturbance structure is re-emphasized. An added advantage of the systematic approach is that indefinitely many further terms in the expansions follow in principle from the pattern established above. In particular, non-parallel-flow effects would emerge as relative corrections of order ϵ^5 at least and are probably even greater than that, from an analogy with Bodonyi & Smith's (1981) conclusions on the Blasius case.

The stability theory and structure above for the infinitesimal disturbance of (2.3) lay the foundation for what follows. For next we wish to consider significantly bigger disturbances. It is found that linear theory continues to hold good to leading order provided the disturbance size δ remains less than $\epsilon^{\frac{7}{3}}(= Re^{-\frac{7}{36}})$, but when δ reaches $O(\epsilon^{\frac{7}{3}})$ a significant new feature comes into play. This is the influence of nonlinear interaction in the streamwise momentum balance within the critical layer III, whereas outside III no major alteration of the leading-order balances described above is implied. Consequently, we move on immediately to the discussion of a disturbance whose size δ is $O(\epsilon^{\frac{7}{3}})$.†

3. Less-small disturbances and the nonlinear critical layer

The suggestion just made requires that we reconsider the Navier–Stokes equations with δ of order $\epsilon^{\frac{7}{3}}$. Since the significant changes in balance are found to occur only within the critical layer III we concentrate mainly on that layer and outside it present only the slightly altered forms of zones II, IV where the effects produced are typical of those outside the critical layer.

3.1. *Outside the critical layer*

Henceforth, periodic travelling-wave solutions are sought dependent only on $\tilde{X} \equiv (X - c\epsilon^{-1}\tau)$ and the appropriately scaled y, rather than on X, τ and scaled y, implying that our concern again lies in tracing the development of the neutral condition but with the disturbance size varying through the current $O(\epsilon^{\frac{7}{3}})$ stage. Here the wavenumber α is kept fixed, at order ϵ^{-5} (cf. (2.10*b*)) for definiteness. So the object becomes to find the position x and wave speed $c = \epsilon\hat{c}$ (cf. (2.10*b*)) at which a disturbance of size $O(\epsilon^{\frac{7}{3}})$ and given wavenumber α is neutrally stable (although the study could equally well be conducted for a fixed frequency disturbance and that would yield the same neutral stability criteria). Here the rather complicated balances governing the stability properties re-assert themselves. Thus in § 2 the relative error in the neutral criteria (2.10*a*) is $O(\epsilon)$ (see (2.5*b*)); therefore the neutral position x is determined only to within $O(\epsilon)$ by (2.10*a*). Accordingly the skin friction λ and the curvature λ_2 have to be expanded in terms of ϵ to enable the disturbance of wavenumber α to be maintained in the neutral state. Conversely, a slow growth of the disturbance could be accommodated by a multiple scales approach with $\partial/\partial\tilde{X}$ being

† The nonlinear critical-layer effect is then much stronger than the effect ($\propto$ (amplitude)^3) of weakly nonlinear stability theory (Stuart 1960; Watson 1960; Pekeris & Shkoller 1967; Reynolds & Potter 1967). For the first effect produces a significant change (see § 5 below) in the neutral stability curve compared with linear theory, whereas the second effect strictly produces only a small change. The two effects are comparable (and weak) only for distances much smaller than $O(\epsilon^{\frac{7}{3}})$.

replaced by $\partial/\partial\tilde{X}+\epsilon\,\partial/\partial\bar{X}+\ldots$, say, where the slow co-ordinate $\bar{X}$ governs the amplitude modulation, and the wavenumber could be considered slightly displaced from its unknown neutral condition. However, let us establish what the new neutral state actually is now, in the spirit of the studies by Benney & Bergeron (1969), Haberman (1972), Brown & Stewartson (1978) and others, leaving consideration of modulations from that state until later. Following the above comments we are led to the expressions

$$\left.\begin{aligned} &\alpha = \epsilon^{-5}\alpha_0, \quad c = \epsilon c_0+\epsilon^2 c_1+\ldots, \quad x = x_0+\epsilon x_1+\ldots, \\ &\lambda = \lambda^{(0)}+\epsilon\lambda^{(1)}+\ldots, \quad \lambda_2 = \lambda_2^{(0)}+\epsilon\lambda_2^{(1)}+\ldots, \quad U_0 = U_{\mathrm{B}}^{(0)}+\epsilon U_{\mathrm{B}}^{(1)}+\ldots \end{aligned}\right\} \tag{3.1a}$$

governing neutral stability, where, for given α_0, the constants $c_j, x_j, \lambda^{(j)}, \lambda_2^{(j)}$ are to be found. The skin-friction and curvature coefficients $\lambda^{(j)}, \lambda_2^{(j)}$ ($\lambda_2^{(0)} < 0$) depend only on the values of x_j in fact, from the prescribed boundary-layer profile $U_{\mathrm{B}}(x, Y)$, which is also expressed in expanded form in (3.1a). All quantities here and below are real.

The flow solution in zone II may be considered first. There, with $\delta = O(\epsilon^{\frac{7}{3}})$ the expansions for the total velocity and pressure are

$$\bar{u} = \epsilon\lambda\tilde{Y}+\epsilon^2\lambda_2\tilde{Y}^2+\epsilon^{\frac{7}{3}}\bar{u}^{(0)}(\tilde{X},\tilde{Y})+\epsilon^{\frac{8}{3}}u_{\mathrm{MF}}(\tilde{Y})+\epsilon^{\frac{10}{3}}\bar{u}^{(1)}(\tilde{X},\tilde{Y})+\ldots, \tag{3.1b}$$

$$\bar{v} = \epsilon^{\frac{13}{3}}\bar{v}^{(0)}(\tilde{X},\tilde{Y})+\epsilon^{\frac{16}{3}}\bar{v}^{(1)}(\tilde{X},\tilde{Y})+\ldots, \tag{3.1c}$$

$$\bar{p} = \epsilon^{\frac{10}{3}}\bar{p}^{(0)}(\tilde{X},\tilde{Y})+\epsilon^{\frac{13}{3}}\bar{p}^{(1)}(\tilde{X},\tilde{Y})+\ldots, \tag{3.1d}$$

mainly in view of (2.2), (2.5a). Also, however, (3.1a) holds, while the mean flow correction $u_{\mathrm{MF}}(\tilde{Y})$ is included in anticipation of the critical-layer correction below. This mean-flow correction has no effect on the major disturbance contributions. From the Navier–Stokes equations the solution satisfying the tangential-flow constraint as $\tilde{Y}\to 0+$ is

$$\bar{u}^{(0)} = 2\lambda^{(0)}A_0\cos\alpha_0\tilde{X}, \quad \bar{v}^{(0)} = 2\alpha_0\lambda^{(0)}A_0\tilde{Y}\sin\alpha_0\tilde{X}, \quad \bar{p}^{(0)} = 2P_0\cos\alpha_0\tilde{X}. \tag{3.2a}$$

Here the unknown constants A_0, P_0 are related by the pressure–displacement rule

$$\lambda^{(0)}c_0A_0 = P_0, \tag{3.2b}$$

akin to (2.8c), from the momentum balance. Similarly we find, in keeping with (2.6b), the next-order forms

$$\left.\begin{aligned} \bar{u}^{(1)} &= 4\lambda_2^{(0)}A_0(\xi+\kappa\ln\xi+\kappa)\cos\alpha_0\tilde{X}+\lambda^{(0)}a(\tilde{X}), \\ \bar{v}^{(1)} &= 2\lambda_2^{(0)}\alpha_0A_0(\xi^2+2\kappa\xi\ln\xi-\kappa^2)\sin\alpha_0\tilde{X}-\frac{1}{\lambda^{(0)}}\frac{\partial\bar{p}^{(1)}}{\partial\tilde{X}}-\lambda^{(0)}\xi a'(\tilde{X}) \\ &\quad -2\alpha_0A_0c_1\sin\alpha_0\tilde{X}, \\ \partial\bar{p}^{(1)}/\partial\tilde{Y} &= 0, \end{aligned}\right\} \tag{3.2c}$$

for $\xi > 0$, where $\xi = \tilde{Y}-\kappa$ and $\kappa \equiv c_0/\lambda^{(0)}$. The additional displacement $-a(\tilde{X})$ and pressure $\bar{p}^{(1)}$ are unknown functions of $\tilde{X}$, but in view of the periodicity requirement they are expressible as Fourier series

$$\left.\begin{aligned} a(\tilde{X}) &= \sum_{n=0}^{\infty}\{a_n\cos n\alpha_0\tilde{X}+\tilde{a}_n\sin n\alpha_0\tilde{X}\}, \\ \bar{p}^{(1)}(\tilde{X}) &= \sum_{n=0}^{\infty}\{p_n\cos n\alpha_0\tilde{X}+\tilde{p}_n\sin n\alpha_0\tilde{X}\}. \end{aligned}\right\} \tag{3.2d}$$

The continuation, below the critical layer III (at $\xi = 0$, $\tilde{Y} = \kappa$), of the solutions in (3.2*c*) requires an examination of the critical-layer properties. Prior to that we note that the zones I, V (as in § 2) reproduce only linear disturbance properties, in a similar vein to (3.2*a*), and so yield the pressure–displacement relations

$$\left.\begin{aligned} P_0 = \alpha_0 A_0, \quad \tilde{p}_1 = \alpha_0 \tilde{a}_1, \\ p_1 = \alpha_0 a_1 + P_0\left[\alpha_0 \int_0^\infty \{U_B^{(0)2} - U_B^{(0)-2}\}\, dY + \frac{\lambda^{(1)}}{\lambda^{(0)}} + \int_0^\infty \frac{\partial U_B^{(1)}}{\partial Y} \frac{dY}{U_B^{(0)}} - \frac{4\lambda_2^{(0)}}{\lambda^{(0)}}\kappa\right], \\ p_n = n\alpha_0 a_n, \quad \tilde{p}_n = n\alpha_0 \tilde{a}_n \quad \text{for} \quad n = 2, 3, 4, \ldots \end{aligned}\right\} \tag{3.2e}$$

akin to (2.8*b*), for the Fourier components of (3.2*a*, *c*, *d*). The forcing terms in the p_1–a_1 relation here arise from interactions like those of Bodonyi & Smith (1981) and from the uncertainty in x in (3.1*a*). The terms with $n = 0$ either give only a small mean-flow correction or are irrelevant.

3.2. *The nonlinear critical layer*

The critical-layer expansions, implied mainly by (3.1*a*–*d*), (3.2*a*–*c*), are now

$$\begin{aligned} \bar{u} = {} & \epsilon c_0 + \epsilon^{\frac{5}{3}}\lambda^{(0)} Y_1 + \epsilon^2[\lambda^{(1)}\kappa + \lambda_2^{(0)}\kappa^2] + 2\epsilon^{\frac{7}{3}}\lambda^{(0)} A_0 \cos\alpha_0 \tilde{X} + \epsilon^{\frac{8}{3}}[\lambda^{(1)} + 2\kappa\lambda_2^{(0)}]\, Y_1 \\ & + \epsilon^3[\lambda^{(2)}\kappa + \lambda_2^{(1)}\kappa^2] + \tfrac{8}{3}\epsilon^{\frac{10}{3}} \ln(\epsilon)\, \lambda_2^{(0)} \kappa A_0 \cos\alpha_0 \tilde{X} + \epsilon^{\frac{10}{3}} \tilde{U}_1(\tilde{X}, \tilde{Y}) + \ldots, \end{aligned} \tag{3.3a}$$

$$\begin{aligned} \bar{v} = {} & 2\epsilon^{\frac{13}{3}} \alpha_0 A_0 c_0 \sin\alpha_0 \tilde{X} + 2\epsilon^5 \alpha_0 \lambda^{(0)} A_0 Y_1 \sin\alpha_0 \tilde{X} + \epsilon^{\frac{16}{3}} \bar{v}^{(1)}|_{\xi=0} + \epsilon^{\frac{17}{3}} \tilde{v}_c(\tilde{X}) \\ & + \tfrac{8}{3}\epsilon^6 \ln(\epsilon)\, \lambda_2^{(0)} \kappa \alpha_0 A_0 Y_1 \sin\alpha_0 \tilde{X} + \epsilon^6 \tilde{V}_1(\tilde{X}, \tilde{Y}) + \ldots, \end{aligned} \tag{3.3b}$$

$$\bar{p} = 2\epsilon^{\frac{10}{3}} P_0 \cos(\alpha_0 \tilde{X}) + \epsilon^{\frac{13}{3}} \bar{p}^{(1)} + \epsilon^{\frac{14}{3}} \tilde{p}_c(\tilde{X}) + \epsilon^5 \tilde{P}_1(\tilde{X}, \tilde{Y}) + \ldots, \tag{3.3c}$$

where Y_1 is $O(1)$: $y = \epsilon^7 \kappa + \epsilon^{\frac{23}{3}} Y_1$. The additional terms of order $\epsilon^{\frac{17}{3}}$ in $\bar{v}$ and $\epsilon^{\frac{14}{3}}$ in $\bar{p}$ here are higher harmonics induced by the added nonlinearity of the disturbance. Indeed, the first non-trivial balances resulting from substitution of (3.3*a*–*c*) into the Navier–Stokes equations give the solutions

$$\tilde{p}_c(\tilde{X}) = 2\tilde{\tilde{p}} \cos(2\alpha_0 X), \quad \tilde{v}_c(\tilde{X}) = 2\left[\lambda^{(0)}\alpha_0 A_0^2 + \frac{2\alpha_0}{\lambda^{(0)}}\tilde{\tilde{p}}\right] \sin 2\alpha_0 \tilde{X}. \tag{3.4}$$

Here $\tilde{\tilde{p}}$ is an undetermined constant. The next non-trivial balances then yield the main equations of the nonlinear critical layer,

$$\partial \tilde{U}_1/\partial \tilde{X} + \partial \tilde{V}_1/\partial Y_1 = 0, \tag{3.5a}$$

$$\lambda^{(0)} Y_1 \frac{\partial \tilde{U}_1}{\partial \tilde{X}} + \lambda^{(0)} \tilde{V}_1 + 2\alpha_0 c_0 A_0 \sin(\alpha_0 \tilde{X}) \frac{\partial \tilde{U}_1}{\partial Y_1} = -\frac{\partial \tilde{P}_1}{\partial \tilde{X}} + \frac{\partial^2 \tilde{U}_1}{\partial Y_1^2}, \tag{3.5b}$$

$$0 = -\frac{\partial \tilde{P}_1}{\partial Y_1}, \tag{3.5c}$$

governing $\tilde{U}_1$, $\tilde{V}_1$, $\tilde{P}_1$. The asymptotic conditions expected are

$$\tilde{U}_1 \sim \lambda_2^{(0)} Y_1^2 + \tilde{\lambda}_+ Y_1 + 4\lambda_2^{(0)} \kappa A_0 \cos(\alpha_0 \tilde{X}) \ln(Y_1) + \tilde{U}_+(\tilde{X}) \quad \text{as} \quad Y_1 \to \infty, \tag{3.6a}$$

$$\tilde{U}_1 \sim \lambda_2^{(0)} Y_1^2 + \tilde{\lambda}_- Y_1 + 4\lambda_2^{(0)} \kappa A_0 \cos(\alpha_0 \tilde{X}) \ln|Y_1| + \tilde{U}_-(\tilde{X}) \quad \text{as} \quad Y_1 \to -\infty, \tag{3.6b}$$

where $\tilde{\lambda}_\pm$ are unknown constants. Here the contributions proportional to Y_1^2 and $\ln Y_1$ effect the match with the basic curvature term (see (2.4*b*), (3.1*a*)) and the logarithm occurring in the fundamental disturbance (3.2*b*), as $\tilde{Y} \to \kappa^+$ in zone II. The same logarithm is expected to arise to arise as $\tilde{Y} \to \kappa-$ but with $\kappa - \tilde{Y}$ replacing $\tilde{Y} - \kappa$. The other contributions, $O(Y_1)$, $O(1)$, are unknown but they reflect the twofold effect of the critical layer and follow from Brown & Stewartson's (1978) modification of the conditions proposed by Benney & Bergeron (1969) and Haberman (1972). First the $O(Y_1)$ terms give a jump $\tilde{\lambda}_+ - \tilde{\lambda}_-$ in the mean vorticity $\partial\tilde{U}_1/\partial Y_1$, and hence a non-zero mean-flow correction of order $\epsilon^{\frac{8}{3}}$ in $\bar{u}$ in zone II outside, as anticipated by (3.1*b*). Therefore $u_{\mathrm{MF}}(\tilde{Y})$ in (3.1*b*) must satisfy the constraint

$$u_{\mathrm{MF}}(\tilde{Y}) \sim u_{\mathrm{MF}}(\kappa) + \tilde{\lambda}_{\pm}(\tilde{Y} - \kappa) \quad \text{as} \quad \tilde{Y} \to \kappa\pm, \tag{3.7}$$

although it has *no* interactive effect on the dominant fundamentals (3.2*a*, *b*). For convenience it is supposed that $u_{\mathrm{MF}}(\kappa) = 0$, although an origin shift in Y_1 allows for a non-zero value of $u_{\mathrm{MF}}(\kappa)$. The mean-flow correction u_{MF} is seen to be *smaller* than the fundamental disturbance $\bar{u}^{(0)}$, in (3.1*b*), a feature which contrasts with the suggestions of Benney & Bergeron (1969), Haberman (1972), Brown & Stewartson (1978) and others. This contrast arises because the classically based approach overlooks the systematic balancing of the disturbance structure. This balancing requires the fundamental disturbance to be *regular* at the critical layer to leading order (i.e in $\bar{u}^{(0)}$ in (3.2*a*)) and to exhibit the logarithmic singularity only at the next order (i.e. in $\bar{u}^{(1)}$ in (3.2*c*)), whereas the classical approach strictly errs by supposing the two orders to be identical. Fortunately, the classical and the systematic approaches can be reconciled on this and most (but not all) other counts when allowance is made for such aberrations. The second effect of the nonlinear critical layer concerns the $O(1)$ contributions in (3.6*a*, *b*). The difference $\tilde{U}_+ - \tilde{U}_-$ determines the induced velocity jump which is emphasized by Brown & Stewartson (1978), is $O(\epsilon^{\frac{10}{3}})$ from (3.3*a*) and must be found in order to allow the $O(\epsilon^{\frac{10}{3}})$ inviscid solution (3.2*c*) for $\bar{u}^{(1)}$ (and hence $\bar{v}^{(1)}$) in zone II to be continued below the critical layer. In the linear critical layer (§ 2) for any disturbance size δ less than $O(\epsilon^{\frac{7}{3}})$ this velocity jump merely involves a phase shift of $-\pi$ in the logarithm in the fundamental. For the current nonlinear critical layer the velocity jump is more complicated, however, and further comments on it will be made shortly.

The transformation

$$\tilde{U}_1 = 4\lambda_2^{(0)}\kappa A_0 \frac{\partial\psi^*}{\partial Y^*}, \quad \tilde{V}_1 = -2\alpha_0\lambda_2^{(0)}(2\kappa A_0)^{\frac{3}{2}}\frac{\partial\psi^*}{\partial X^*}, \quad \tilde{P}_1 = 2\lambda^{(0)}\lambda_2^{(0)}(2\kappa A_0)^{\frac{3}{2}}\, P^*(X^*),$$
$$\tilde{X} = \alpha_0^{-1}X^*, \quad Y_1 = (2\kappa A_0)^{\frac{1}{2}}\, Y^* \tag{3.8}$$

now yields, from (3.5*a*–*c*), (3.6*a*, *b*), the nonlinear critical-layer equation and boundary conditions

$$Y^* \frac{\partial^3\psi^*}{\partial X^*\,\partial Y^{*2}} + \frac{\partial^3\psi^*}{\partial Y^{*3}}\sin X^* = \gamma_c \frac{\partial^4\psi^*}{\partial Y^{*4}}, \tag{3.9a}$$

with, as $Y^* \to \pm\infty$,

$$\frac{\partial\psi^*}{\partial Y^*} \sim \begin{cases} \tfrac{1}{2}Y^{*2} + 2H_+\,Y^* + \cos(X^*)\ln Y^* + U_+^*(X^*), & (3.9b)\\ \tfrac{1}{2}Y^{*2} + 2H_-Y^* + \cos(X^*)\ln|Y^*| + U_-^*(X^*), & (3.9c)\end{cases}$$

and the requirement of periodicity of 2π in X^*. Here

$$H_\pm \equiv \frac{\tilde{\lambda}_\pm}{4\lambda_2^{(0)}(2\kappa A_0)^{\frac{1}{2}}}, \quad U_\pm^*(X^*) \equiv \frac{\tilde{U}_\pm(\tilde{X})}{4\lambda_2^{(0)}\kappa A_0} \tag{3.10a}$$

are to be found, or at least the jump quantities $H_+ - H_-$, $U_+^* - U_-^*$, while

$$\gamma_c^{-1} = \lambda^{(0)}\alpha_0(2\kappa A_0)^{\frac{3}{2}} \tag{3.10b}$$

is an $O(1)$ parameter describing effectively the size δ of the fundamental disturbance relative to the present $O(\epsilon^{\frac{7}{3}})$ scale. The 'nonlinear' aspect here is the second term in (3.9a), from the inertial interaction between the leading-order regular fundamental ($\propto \sin X^*$) and the influence ($\propto \partial^3\psi^*/\partial Y^{*3}$) of the basic flow curvature and the higher-order fundamental. When γ_c becomes large, the effect of the interaction diminishes, as does the disturbance size, so that (3.9a) then tends to Airy's equation, which produces a phase jump of $-\pi$. When γ_c is not large the critical-layer effect from (3.9a–c) is less simple. The problem (3.9a–c) is essentially that of Haberman (1972), but with the modification, suggested by Brown & Stewartson (1978), that the velocity jump is not assumed to be monochromatic. Brown & Stewartson's (1978) analysis for $\gamma_c \gg 1$ showed that $U_+^* - U_-^*$ does not have a simple wave form in general, a conclusion that could also be extracted from Haberman's (1976) study. Accordingly the concept of a phase shift across the critical layer needs to be amended (but not rejected – cf. Brown & Stewartson 1978), and some doubts are raised about Haberman's (1972) numerical solution, since a monochromatic velocity jump was assumed. On the other hand, since periodicity of 2π in X^* is expected, it seems justifiable to Fourier-decompose the velocity jump as

$$U_+^* - U_-^* = \sum_{n=0}^{\infty} \{F_n \cos nX^* + \tilde{F}_n \sin nX^*\}, \tag{3.11a}$$

where the constants $F_n, \tilde{F}_n$ ($n \geqslant 0$) are unknown. So now if we define the phase jump ϕ to be

$$\phi = \tilde{F}_1 = \pi^{-1}\int_0^{2\pi} (U_+^* - U_-^*) \sin X^* \, dX^*, \tag{3.11b}$$

then the value of ϕ has the significance pointed out in § 3.3.

3.3. *The phase shift*

The velocity jump (3.11a) and transformation (3.8) imply that below the critical layer the solutions (3.2c) continue to hold in zone II provided that $\ln \xi$, $a(\tilde{X})$ are replaced as follows for $\xi < 0$:

$$\ln \xi \to \ln |\xi|, \quad a(\tilde{X}) \to a(\tilde{X}) - 4\lambda_2^{(0)}\kappa A_0(U_+^* - U_-^*)/\lambda^{(0)}. \tag{3.12}$$

Then again, the solution in zone II needs to be reconciled with the wall-layer solution as $\xi \to -\kappa$, $\tilde{Y} \to 0+$. In the wall layer IV

$$\left.\begin{aligned}
\bar{u} &= \epsilon^2\lambda^{(0)}Z + \epsilon^{\frac{7}{3}}[\cos \alpha_0 \tilde{X} - e^{-m_1 Z}\cos(\alpha_0\tilde{X} - m_2 Z)]\,\frac{2\alpha_0 P_0}{\beta_0} + \dots,\\
\bar{v} &= \epsilon^{\frac{16}{3}}[\sin(\alpha_0\tilde{X} + \tfrac{1}{4}\pi) - \beta_0^{\frac{1}{2}} Z \sin \alpha_0\tilde{X} - e^{-m_1 Z}\sin(\alpha_0\tilde{X} + \tfrac{1}{4}\pi - m_2 Z)]\,\frac{2\alpha_0^2 P_0}{\beta_0^{\frac{3}{2}}} + \dots,\\
\bar{p} &= 2\epsilon^{\frac{10}{3}} P_0 \cos \alpha_0 \tilde{X} + \dots,
\end{aligned}\right\} \tag{3.13}$$

as suggested by (2.5a) with (2.2) and $\delta = O(\epsilon^{\frac{7}{3}})$, where $m_1 = -m_2 = (\frac{1}{2}\beta_0)^{\frac{1}{2}}$, with $\beta_0 = \alpha_0 c_0$. The wall layer flow *only just* remains linear, by a relative amount $O(\epsilon^{\frac{1}{3}})$ from (3.13). More comments on this are given in § 5. Joining of (3.13) with the inviscid solution outside therefore demands the condition (as in (2.8a))

$$\bar{v}^{(1)}|_{\tilde{Y}=0+} = 2\alpha_0^{\frac{1}{2}} c_0^{-\frac{3}{2}} P_0 \sin(\alpha_0\tilde{X} + \tfrac{1}{4}\pi) \tag{3.14}$$

for zone II. From (3.14), (3.2e), with the replacements (3.12) applied in (3.2c), the higher harmonic components give only the relations

$$p_n = -\frac{4n}{n-1}\lambda_2^{(0)}\frac{c_0^2}{\lambda^{(0)}}A_0 F_n, \quad \tilde{p}_n = -\frac{4n}{n-1}\lambda_2^{(0)}\frac{c_0^2}{\lambda^{(0)}}A_0 \tilde{F}_n \tag{3.15a}$$

for $n \neq 1$, which determine the forced higher harmonics in zone II and elsewhere outside the critical layer. Likewise, the $\sin\alpha_0\tilde{X}$ components in (3.14) give, from (3.2e),

$$2c_1 = -4\lambda_2^{(0)}\kappa^2(\ln\kappa - F_1) - (2\alpha_0)^{\frac{1}{2}}c_0^{-\frac{3}{2}} + \frac{\alpha_0}{\lambda^{(0)}}\left[\alpha_0\int_0^\infty (U_{\mathrm{B}}^{(0)2} - U_{\mathrm{B}}^{(0)-2})\,dY \right.$$
$$\left. + \lambda^{(1)}/\lambda^{(0)} + \int_0^\infty \frac{\partial U_{\mathrm{B}}^{(1)}}{\partial Y}\frac{dY}{U_{\mathrm{B}}^{(0)}} - 4\lambda_2^{(0)}\kappa/\lambda^{(0)}\right], \tag{3.15b}$$

which merely sets a first condition on the uncertainty in x and the wave-speed correction c_1 of (3.1a).

By contrast, from the fundamental $\cos\alpha_0\tilde{X}$ contributions in (3.14) we obtain what amounts to a resonance (cf. (3.15a)) or a solvability condition on the dominant disturbance amplitude, namely

$$-4\alpha_0\lambda_2^{(0)}\frac{c_0^2}{\lambda^{(0)2}}A_0\tilde{F}_1 = 2^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}c_0^{-\frac{3}{2}}P_c, \tag{3.15c}$$

on use of (3.2b,e). Here (3.15c) leaves the amplitude corrections $p_1, \tilde{p}_1, a_1, \tilde{a}_1$ remaining arbitrary thus far, and serves to illustrate again the balancing of the phase shifts $\tilde{F}_1$ ($= \phi$) and $\frac{1}{4}\pi$ produced respectively by the critical layer and wall layer. Further, from (3.15c), (3.2b,e) it follows that the neutral stability criteria (2.10a) continue to hold provided only that $-\pi$ there is replaced by the new phase shift ϕ, to give

$$\alpha_0 = \frac{\lambda^{(0)1\frac{1}{6}}}{2^{\frac{1}{2}}(\phi\lambda_2^{(0)})^{\frac{1}{3}}}, \quad \beta_0 = \frac{\lambda^{(0)\frac{8}{3}}}{2(\phi\lambda_2^{(0)})^{\frac{2}{3}}}, \quad c_0 = \frac{\lambda^{(0)\frac{5}{6}}}{2^{\frac{1}{2}}(\phi\lambda_2^{(0)})^{\frac{1}{3}}}. \tag{3.16}$$

The direct step from (2.10a) to (3.16) was anticipated by Haberman (1972), although without the explicit determination of the induced higher harmonics of (3.2c,d), (3.15a), the perturbation in wave speed, the smallness of the mean-flow correction or the smallness of the wavenumber. Again, the demonstration above, that although these higher harmonics are of quite significant amplitude they still have no influence on the neutral stability properties (3.16), proves very helpful in the theory of part 2.

3.4. *The nonlinear critical-layer calculation*

Because of the neutral criteria (3.16), and contrary to Brown & Stewartson's suggestions, the unknown phase shift ϕ of (3.11b) remains as the most significant overall effect of the nonlinear critical layer. Like the entire critical-layer solution the value of ϕ depends only on the value of γ_c in (3.9a), although it can be related to the vorticity jump by the result

$$\phi = 4\gamma_c(H_+ - H_-), \tag{3.17}$$

of Haberman (1972), which still holds (by double integration of (3.9a)) despite the corrected form (3.11a) of the velocity jump. Therefore only the solution of the problem (3.9a–c) now needs to be determined.

In view of the slight doubts raised earlier about Haberman's (1972) numerical solution, an independent numerical treatment of (3.9a–c) was decided upon. In

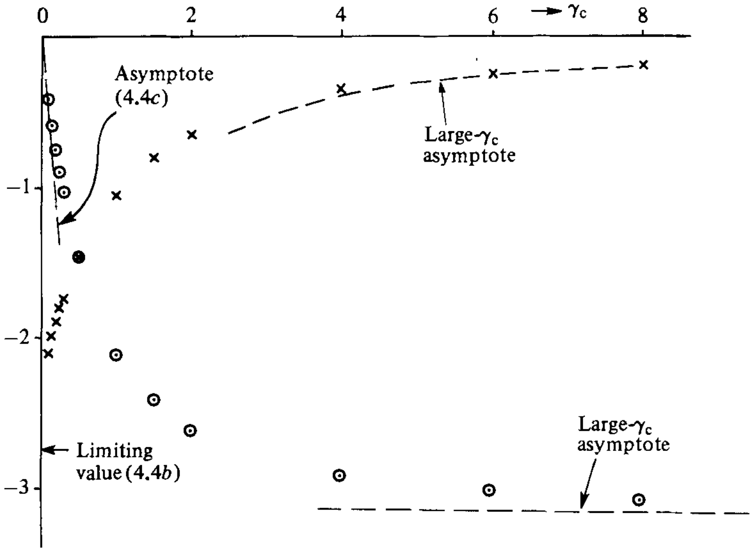

FIGURE 2. Calculated results for the phase shift ϕ (shown by $\odot$) (see (3.11a, b)) and the vorticity jump $2(H_+ - H_-)$ (shown by $\times$) from the solution of (3.9a–c) for various values of γ_c. Also shown are the asymptotes for $\gamma_c \to 0+$ (taken from (4.4b–d)) and $\gamma_c \to \infty$.

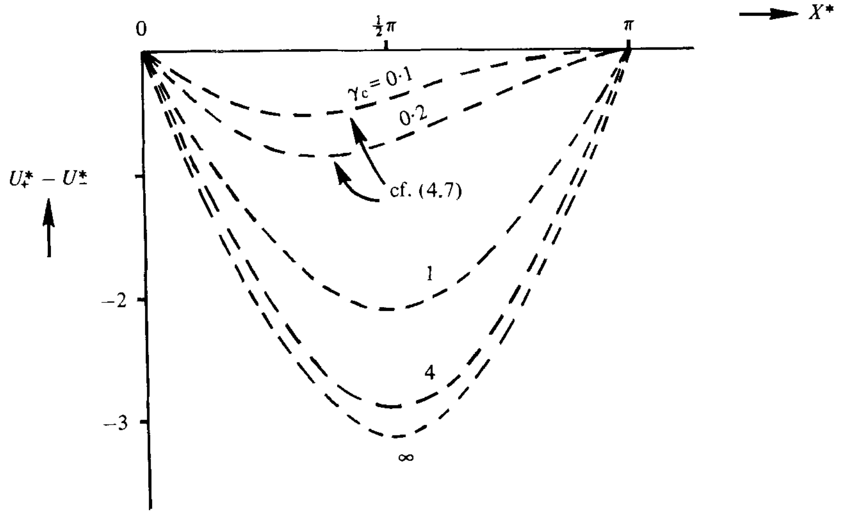

FIGURE 3. Numerical solutions for the velocity jump $U_+^* - U_-^*$ of (3.9a–c) as functions of X^* for various values of γ_c. The velocity jump is odd about $X^* = \pi$ in each case. The result for $\gamma_c = \infty$ stems from the linear theory, which gives the jump $-\pi \sin X^*$ from § 2.

summary, our treatment assumed a Fourier-series solution for $\partial^2\psi^*/\partial Y^{*2}$, which was substituted into (3.9a) to yield a coupled infinite set of ordinary differential equations for the Fourier coefficients. This set, when truncated, was solved by uniform central differencing, iterating and, if necessary, using relaxation. Tests were carried out on the effects of the step sizes (typically 0·02 to 0·2) in Y^*, of the number of terms (typically 10–30) retained in the series truncation, and of the iterative convergence test; and as a result we believe the results to be satisfactory. After the calculations were done, Dr P. Huerre (private communication 1980) kindly informed us that he too had

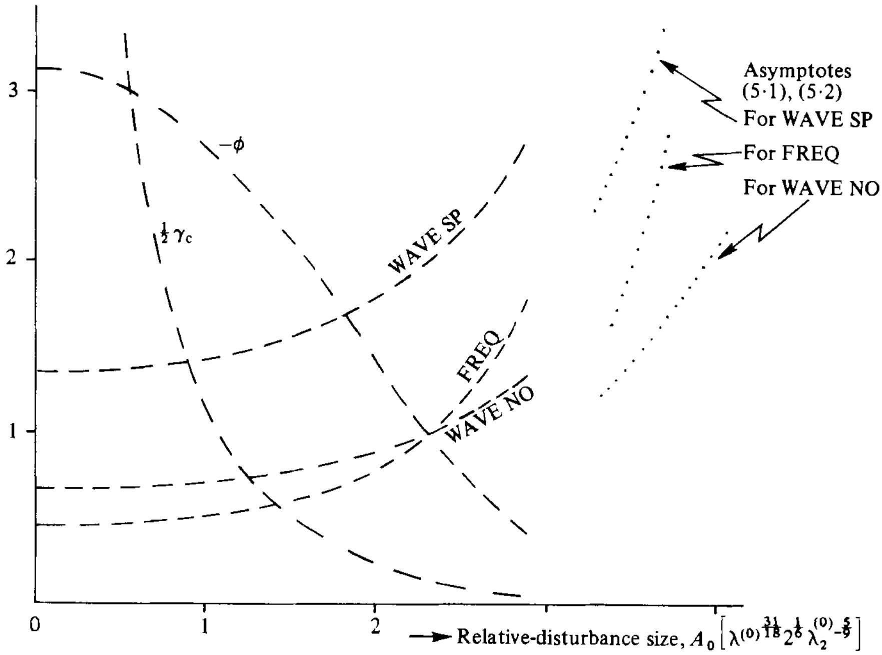

FIGURE 4. The dependence of the neutral wavenumber, wave speed and frequency on the relative disturbance size A_0. Here WAVE SP stands for $2[8\lambda_2^{(0)2}\lambda^{(0)-5}]^{\frac{1}{9}}c_0$. WAVE NO stands for $[8\lambda_2^{(0)2}\lambda^{(0)-11}]^{\frac{1}{9}}\alpha_0$, and FREQ stands for $[8\lambda_2^{(0)2}\lambda^{(0)-8}]^{\frac{1}{9}}\beta_0$ (see (3.16)). Also shown are the negative phase shift $-\phi$ and the variation of γ_c, versus A_0.

recalculated solutions, and his and our results are in close agreement (see also below). The calculated properties of (3.9*a–c*) as γ_c varies are summarized in figure 2, which shows the dependence of the phase shift ϕ and vorticity jump $2(H_+ - H_-)$ on γ_c and in figure 3, which shows the velocity jump of (3.11*a*). Our results for ϕ and $2(H_+ - H_-)$ seem to agree at least graphically with Haberman's (1972) calculations, as do Dr Huerre's. Further, the approach towards the earlier classical linear results, of zero vorticity jump and a phase shift of $-\pi$ when $\gamma_c \to \infty$, appears to be verified as γ_c increases; the analytical solution for $\gamma_c \gg 1$ has been studied by Haberman (1976) and Brown & Stewartson (1978), and is in line with our calculated results. Given ϕ as a function of γ_c, (3.16) enables the dependence of the neutral wavenumber, wave speed and frequency on the relative disturbance size given by A_0 in (3.10*b*) to be worked out; this is presented in figure 4. As the present $O(\epsilon^{\frac{7}{9}})$ disturbance increases in size γ_c falls, the negative phase shift $-\phi$ decreases, and the vorticity jump increases. So the neutral wavelength then shortens, the neutral frequency increases, while the critical layer at $\tilde{Y} = \tilde{Y}_c = \kappa$ is pushed further from the wall, since c_0 increases with decreasing γ_c. The solution properties in the limit as $\gamma_c \to 0$ are especially worthy of study next because the suggestion from the numerical results of figures 2–4 is that this limit corresponds to an enhanced size of disturbance, an attractive prospect on both physical and analytical grounds.

4. The increasingly nonlinear critical layer when $\gamma_c \ll 1$

Although the solution of (3.9a–c) for γ_c small has been examined already by Haberman (1972) and Brown & Stewartson (1978), we present an account here because our method and conclusions differ from theirs in a number of respects. Moreover the account produces some guidelines for the subsequent work in part 2.

When γ_c is small an expansion of the form

$$\psi^* = \psi_0 + \gamma_c \psi_1 + \dots \tag{4.1}$$

is suggested first, for general $O(1)$ values of Y^*. From (3.9a), therefore, the viscous forces are negligible to leading order in γ_c and so

$$\partial^2\psi_0/\partial Y^{*2} = K(\eta), \quad \eta = \tfrac{1}{2}Y^{*2} + \cos X^*, \tag{4.2a}$$

expressing conservation of the unknown vorticity $K(\eta)$ along the main streamlines $\eta =$ const. of the total flow. These streamlines give the Kelvin cat's-eye formation, and are closed for $\eta < 1$. So to avoid a singularity arising in the higher-order terms we have within the cat's-eye

$$K(\eta) = N_0 \quad \text{for} \quad \eta < 1 \tag{4.2b}$$

(the Prandtl–Batchelor theorem), where N_0 is a constant to be determined. Outside the eye, for $\eta > 1$, the vorticity variation is fixed almost uniquely by the next-order term ψ_1, for which the periodicity requirement demands that

$$\frac{d}{d\eta}\left\{K'(\eta)\int_0^{2\pi} 2^{\frac{1}{2}}(\eta - \cos X^*)^{\frac{1}{2}}\,dX^*\right\} = 0.$$

Then integration with respect to η and use of the outer constraints (3.9b, c), which impose $Y^{*-1}K(\eta) \to 1$ for $\eta \to \infty$, implies

$$K'(\eta) = \pm 2\pi\Big/\int_0^{2\pi} 2^{\frac{1}{2}}(\eta - \cos X^*)^{\frac{1}{2}}\,dX^* \tag{4.3a}$$

for $Y^* \gtrless \pm 2^{\frac{1}{2}}(1-\cos X^*)^{\frac{1}{2}}$, above and below the eye, respectively. Because of the discontinuity in the vorticity gradient at $\eta = 1^{\pm}$ implied by (4.3a) and (4.2b), thin viscous layers are inevitable near $\eta = 1$, and these will be discussed shortly. To proceed further requires a boundary condition on $K(\eta)$. Benney & Bergeron (1969) conjectured that the condition could be one of negligible vorticity jump across the critical layer, i.e. that the values of $K(\eta) - Y^*$ as $Y^* \to \pm\infty$ should be identical. However, Haberman (1972) proposed instead the condition of continuity of the vorticity $K(\eta)$ at the edges of the cat's eye, $\eta = 1^{\pm}$, a condition suggested strongly by his and our numerical solutions in fact. The matter was then settled by Brown & Stewartson (1978), whose study of the thin viscous layers near the edges of the eye (see below, however) proved the Haberman (1972) option to the correct one. In consequence, the integration of (4.3a) gives

$$K(\eta) = N_0 + \int_1^{\eta} K'(\bar{\eta})\,d\bar{\eta} \tag{4.3b}$$

for $\eta > 1$. So since $K'(\eta)$ is an odd function of Y^* outside the cat's eye from (4.3*a*), the finite parts of (4.3*b*) as $Y^* \to \pm\infty$ ($\eta \to \infty$) immediately yield the results

$$N_0 = H_+ + H_-, \tag{4.4a}$$

$$2(H^+ - H^-) = 2\int_1^\infty \{|K'(\eta)| - (2\eta)^{-\frac{1}{2}}\}\,d\eta - 8^{\frac{1}{2}} \quad [\equiv \tfrac{1}{2}C^{(1)}], \tag{4.4b}$$

to leading order in γ_c, from (3.9*b*, *c*). Here, for $\gamma_c \to 0$, (4.4*a*) predicts the vorticity within the cat's eye to be the average of the mean vorticity corrections induced at the outer extremes of the critical layer, while the limit (4.4*b*) for the mean vorticity jump yields the limiting behaviour

$$\phi \sim C^{(1)}\gamma_c \quad \text{as} \quad \gamma_c \to 0, \tag{4.4c}$$

from (3.11*b*). The jump formulae (4.4*b*, *c*) agree with Haberman's (1972, equations (4.24), (4.26)), although we find

$$\tfrac{1}{4}C^{(1)} = -1{\cdot}379 \tag{4.4d}$$

from (4.4*b*), in place of his estimate $-1{\cdot}05$; the function L in his (4.15) is not independent of his ξ; and the working in his (4.16)–(4.23) needs modification as it assumes only part of the complete velocity jump (our (3.11*a*)) as mentioned before. Again, we note that knowing the vorticity alone is enough to fix the phase jump (4.4*c*); the velocity is not required.

The discontinuities in the vorticity gradients $K'(\eta)$ above near the edges of the eye are smoothed out in thin, upper and lower, viscous layers astride

$$Y^* = \pm 2^{\frac{1}{2}}(1-\cos X^*)^{\frac{1}{2}}.$$

For the upper viscous layer $Y^* = 2^{\frac{1}{2}}(1-\cos X^*)^{\frac{1}{2}} + \gamma_c^{\frac{1}{2}} z$, where z is $O(1)$, and

$$\partial^2\psi^*/\partial Y^{*2} = N_0 + \gamma_c^{\frac{1}{2}}\,\Omega(X^*, z) + \ldots, \tag{4.5a}$$

so that (3.9*a*) yields the equation

$$2^{\frac{1}{2}}(1-\cos X)^{\frac{1}{2}}\frac{\partial\Omega}{\partial X^*} - \frac{2\sin X^*}{2^{\frac{1}{2}}(1-\cos X^*)^{\frac{1}{2}}}\frac{\partial\Omega}{\partial z} = \frac{\partial^2\Omega}{\partial z^2} \tag{4.5b}$$

for Ω. The external inviscid solutions (4.2*a*, *b*) with (4.3*a*, *b*) then lead to the boundary conditions

$$\left.\begin{aligned} &\Omega \sim 2\mathscr{L}_2(X^*)z + \mathscr{L}_1^+ && \text{as} \quad z \to \infty, \\ &\Omega \to \mathscr{L}_1^- && \text{as} \quad z \to -\infty, \end{aligned}\right\} \tag{4.5c}$$

for $0 < X^* < 2\pi$, where $\mathscr{L}_2(X^*) = \frac{1}{8} \times 2^{\frac{1}{2}}\pi(1-\cos X^*)^{\frac{1}{2}}$ from (4.3*a*), and $\mathscr{L}_1^\pm$ are unknown constants. Similar features apply in the lower viscous layer, while a co-ordinate transformation of (4.5*b*) yields the heat-conduction equation whose solution, with the boundary conditions (4.5*c*) and the requirement of periodicity, was obtained by Brown & Stewartson (1978). The following alternative approach produces the major part of the velocity jump across the viscous layers more simply, however.

Integrating (4.5*b*) with respect to z gives the equation

$$2^{\frac{1}{2}}(1-\cos X^*)^{\frac{1}{2}}\frac{\partial U^*}{\partial X^*} + \frac{\sin X^*}{2^{\frac{1}{2}}(1-\cos X^*)^{\frac{1}{2}}}\left(U^* - z\frac{\partial U^*}{\partial z}\right) = -G(X^*) + \frac{\partial^2 U^*}{\partial z^2} \tag{4.6a}$$

for the effective velocity $U^* \equiv \int \Omega\, dz$, where $G(X^*)$ is an unknown function of X^* representing the pressure gradient. Similarly, integration of (4.5*c*) suggests that

$$\left.\begin{aligned} U^* &\sim \mathscr{L}_2(X^*)\,z^2 + \mathscr{L}_1^+ z + \mathscr{L}_0^+(X^*) \quad \text{as} \quad z \to +\infty, \\ U^* &\sim \mathscr{L}_1^- z + \mathscr{L}_0^-(X^*) \qquad\qquad\quad\;\; \text{as} \quad z \to -\infty, \end{aligned}\right\} \tag{4.6b}$$

if we assume a fast-enough approach of Ω to its asymptotes in (4.5*c*). On substitution of (4.6*b*) into (4.6*a*) the expression for $\mathscr{L}_2$ given just after (4.5*c*) is verified, at order z^2 in (4.6*a*), the constancy of $\mathscr{L}_1^{\pm}$ is verified at order z, while the zeroth-order terms yield the equation

$$\frac{d}{dX^*}[2^{\frac{1}{2}}(1-\cos X^*)^{\frac{1}{2}}(\mathscr{L}_0^+ - \mathscr{L}_0^-)] = 2\mathscr{L}_2(X^*) \tag{4.6c}$$

for $\mathscr{L}_0^+ - \mathscr{L}_0^-$. Hence, expecting an odd dependence on $X^* - \pi$, we obtain

$$\mathscr{L}_0^+ - \mathscr{L}_0^- = -\tfrac{1}{2}\pi \cot \tfrac{1}{2}X^*, \tag{4.6d}$$

which determines the effective velocity jump $\mathscr{L}_0^+ - \mathscr{L}_0^-$. The same jump occurs across the lower viscous layer, and so the jump of the velocity $\partial\psi^*/\partial Y^*$ across the two viscous layers is

$$-\gamma_c \pi \cot \tfrac{1}{2}X^*, \tag{4.7}$$

to leading order, in view of (4.5*a*).

The prediction (4.7) is now supported by Brown & Stewartson's (1980*b*) amendments of their (1978) formula (3.30), following a correspondence between the last-named authors and ourselves. Three final points on the nonlinear critical-layer solution for small γ_c seem worth making. Firstly, there is the simplicity of the argument in (4.6*a*)–(4.7). It gives the dominant part of the velocity jump everywhere, except in thin $O(\gamma_c^{\frac{1}{2}})$ regions near $X^* = 0, 2\pi$; the more involved Brown–Stewartson approach is only necessary if the next-order correction is sought. The present argument proves crucial for the new, strongly nonlinear, critical-layer structure described in § 5 and to be examined in part 2. Secondly, and in consequence, the relative errors in the formulae (4.4*b*, *c*), (4.7) are of order $\gamma_c^{\frac{1}{2}}$. Thirdly, the phase shift associated with the viscous velocity jump (4.7) alone is $-2\gamma_c\pi$ (or $-\gamma_c\pi$ across each of the viscous layers), by formal analogy with (3.11*b*), and this provides only part of the total phase shift of (4.4*c*). The remainder comes from the inviscid zone outside the cat's eye. For integration of $\partial^2\psi_1/\partial Y^{*2}$, with respect to Y^* from $Y^* = 2^{\frac{1}{2}}(1-\cos X^*)^{\frac{1}{2}}$ to Y^*, to give the velocity, followed by multiplication by $\pi^{-1}\sin X^*$, integration with respect to X^* from $X^* = 0$ to $X^* = 2\pi$, interchanging the orders of integration in the resultant triple integral and taking the limit $Y^* \to \infty$, gives a phase shift of $\gamma_c(\frac{1}{2}C^{(1)} + \pi)$ through the upper inviscid zone, on use of (4.3*a*). The same phase shift is induced through the lower inviscid zone. Since the phase shift across the inviscid eye is zero to order γ_c, the total phase shift of (4.4*c*) is therefore recovered.

Comparisons between the predictions (4.4*b*–*d*) for $\gamma_c \to 0$ and the calculated solutions of § 3 are given in figures 2, 3, and, bearing in mind the $O(\gamma_c^{\frac{1}{2}})$ relative error, we believe confidence in those predictions to be not unreasonable. The implications of the predictions as far as the influence of disturbance size is concerned will be considered next.

5. The implications for bigger disturbances

From (4.4c) and (3.16) the behaviours

$$\alpha_0 \sim \left[\frac{\lambda^{(0)\frac{11}{6}}}{2^{\frac{1}{2}}(C^{(1)}\lambda_2^{(0)})^{\frac{1}{3}}}\right]\gamma_c^{-\frac{1}{3}}, \quad \beta_0 \sim \left[\frac{\lambda^{(0)\frac{8}{3}}}{2(C^{(1)}\lambda_2^{(0)})^{\frac{2}{3}}}\right]\gamma_c^{-\frac{2}{3}}, \quad c_0 \sim \left[\frac{\lambda^{(0)\frac{5}{6}}}{2^{\frac{1}{2}}(C^{(1)}\lambda_2^{(0)})^{\frac{1}{3}}}\right]\gamma_c^{-\frac{1}{3}} \tag{5.1}$$

are predicted when $\gamma_c \to 0$. Comparisons between (5.1) and the full solution of the nonlinear critical layer are given in figures 2, 3; and again some support for (4.4c, d) as well as (5.1) is thereby provided. Equally important is the asymptotic form

$$A_0 \sim \left[\frac{(C^{(1)}\gamma_2^{(0)})^{\frac{5}{9}}}{2^{\frac{1}{6}}\lambda^{(0)\frac{31}{18}}}\right]\gamma_c^{-\frac{1}{9}} \quad \text{as} \quad \gamma_c \to 0, \tag{5.2}$$

implied by (4.4c), (5.1) and the definition (3.10b). This confirms the expected increase of disturbance size corresponding to decreasing γ_c, since essentially A_0 denotes the amplitude, relative to the current $O(\epsilon^{\frac{7}{3}})$ size of scale, of the fundamental disturbance in zones II, IV, by (3.16), (3.2a), (3.3a) and (3.13), while (5.1) shows that simultaneously the wavelength and wave speed respectively decrease and increase sharply. Therefore, as the effective disturbance size $\delta = A_0\epsilon^{\frac{7}{3}}$ is increased indefinitely during the current stage ($\delta = O(\epsilon^{\frac{7}{3}})$), the whole neutrally stable structure shrinks because the lateral extent of the outermost zone V of potential flow is also proportional to α_0^{-1}. Eventually then, for $\delta \gg \epsilon^{\frac{7}{3}}$, $A_0 \gg 1$, a new structure must come into operation beyond the current stage. The conclusion we draw from the behaviours (5.1), (5.2) is that the new structure will probably appear when the disturbance size δ rises to $O(\epsilon^2 = Re^{-\frac{1}{6}})$. For then formally A_0 must rise to $\epsilon^{-\frac{1}{3}}$, and so γ_c falls to $O(\epsilon^3)$ from (5.2), leading to increases of both α_0 and c_0 to $O(\epsilon)$ from (5.1). In view of (3.1a), which with (5.1) gives the unscaled wavenumber and wave speed of the neutral disturbance, the wavelength would become $O(\epsilon^6) = O(Re^{-\frac{1}{2}})$ then, and the wave speed would reach $O(1)$, so that the critical layer is pushed toward an $O(Re^{-\frac{1}{2}})$ distance from the wall. At that stage the fundamental assumptions of the present work become invalid, of course, owing to the new positioning of the critical layer in the midst of the boundary layer, as well as the equal importance to be set upon lateral and streamwise variations of the fundamental disturbance in the main boundary layer as the outer potential-flow zone becomes coincident with it. In other words, when δ is increased to $O(Re^{-\frac{1}{6}})$ we approach exactly the classical Rayleigh scalings where the majority of the inviscid characteristics of the disturbance are controlled over length scales comparable with the typical lateral scale of the basic flow (figure 5).

In some ways it is comforting to find these more readily understandable classical scalings emerging as we increase the disturbance size δ from its previous rather bizarre order $Re^{-\frac{7}{36}}$ to the suggested new order $Re^{-\frac{1}{6}}$. However, since this new order is $R^{-\frac{1}{3}}$, where R ($= O(Re^{\frac{1}{2}})$) is the Reynolds number based on the boundary-layer thickness, we produce a condition significantly *different* from the classically based one (e.g. Benney & Bergeron 1969; Haberman 1972) that δ be $O(R^{-\frac{2}{3}})$ for a nonlinear critical layer on the Rayleigh scalings. There are some other features indicated that may also cause certain reservations about classically based approaches. Thus, for example, before the classically based stage $\delta = O(Re^{-\frac{1}{3}})$ is reached, the stage $\delta = O(Re^{-\frac{7}{36}})$ that produces the classical nonlinear critical-layer balance of (3.9a) has already been passed

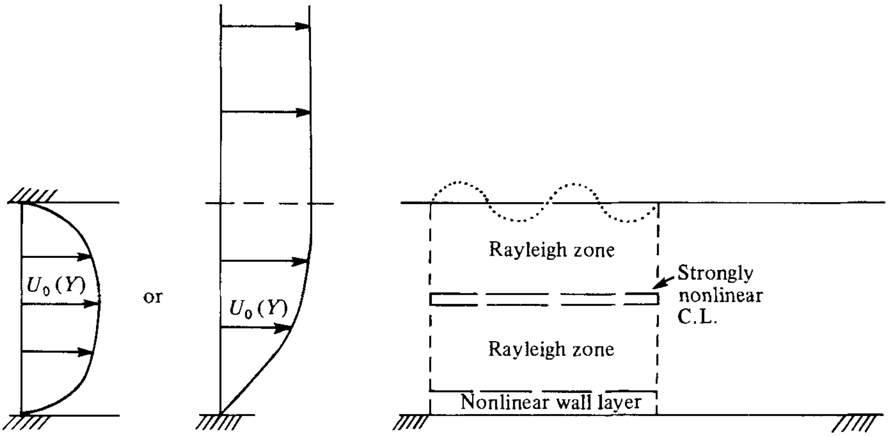

FIGURE 5. Sketch (not to scale) of the relatively compressed structure induced by a travelling-wave disturbance of increased amplitude $\delta = O(Re^{-\frac{1}{6}})$ in a boundary-layer or channel flow with velocity profile $U_0(Y)$, according to the inferences drawn in §5. The form here is to be compared with the form of figure 1 holding for smaller disturbances.

through. Again, it can be seen from (3.3*a*–*c*) that within the critical layer the lateral pressure gradient $\partial\overline{p}/\partial Y^*$ contains some contributions G, say (from $-\tilde{U}_1\, \partial\tilde{V}_1/\partial\tilde{X}$ for instance), proportional to $\kappa^3 A_0^3 \alpha_0^2$ and of order ϵ^{12}. While those contributions are negligible in the dominant balance (3.5*b*, *c*) of the nonlinear critical layer of §§ 3–4, they cannot be so neglected when δ is increased to order $Re^{-\frac{1}{6}}$, as the dependence on $\kappa^3 A_0^3 \alpha_0^2$ then increases G to order ϵ^6, from (5.1), (5.2). This coincides with the order ϵ^6 implied for the pressure $\overline{p}$ due to the $O(\gamma_c)$ perturbation of $\tilde{P}_1$ in (3.8) when the asymptotics of § 4 are applied, by use of (3.3*c*), (3.8) and (5.1), (5.2). As a consequence, precisely when the classical stage of disturbance is reached, the classical assumption of the pressure force being dependent only on the streamwise location becomes invalid within the critical-layer balance. Now consider the wall layer IV. There, the fundamental disturbance in the total velocity $\overline{u}$ of (3.13) is generally a small fraction $O(\alpha_0^2 A_0 \beta_0^{-1})$ of the basic flow velocity. However, from (5.1), (5.2) this fraction will become $O(1)$ as we increase A_0 to $O(\epsilon^{-\frac{1}{3}})$ in line with the increase of δ to $O(Re^{-\frac{1}{6}})$. Hence nonlinearity within the wall layer seems inevitable when we reach the classical Rayleigh stage. Furthermore, the effective wall shear stress $\partial\overline{u}/\partial Y(x, 0)$ due to the basic flow is $O(1)$ throughout, whereas that of the fundamental is $O(\epsilon^{\frac{1}{3}}\alpha_0^2 A_0 \beta_0^{-\frac{1}{2}})$ from (3.13), (3.2*e*). Hence (5.1), (5.2) show formally that the approach to the new size $\delta = O(Re^{-\frac{1}{6}})$ causes the contribution of the fundamental to exceed greatly that of the basic flow, and near the wall a predominantly oscillatory motion, including flow reversal, is indicated. All the features just described tend to be overlooked in the classically based approach, and raise questions about the correct nonlinear structure of the disturbed flow, especially the necessarily novel form to be acquired by the nonlinear critical layer, when the disturbance size is increased to $O(Re^{-\frac{1}{6}})$ or $O(R^{-\frac{1}{3}})$. These questions will be addressed in part 2.

Meanwhile, a final comment on the present study seems appropriate. Throughout, our objective has been quite specific: to follow the dependence of the neutral stability criterion, along the upper branch, on the size of the mainly monochromatic disturbance imposed, and to find the corresponding balance or structure controlling the neutral disturbance. It has also been limited to an attached boundary layer as the basic flow.

Nevertheless, it seems not unreasonable to hope that as a consequence many more physically significant aspects of the stability of streaming flows may prove analysable along rational lines. Certainly, the structural approach used seems to help to emphasize the delicacy of the stability properties. In particular, it unveils an abrupt change in structure and scale as the disturbance size increases just slightly, from $O(Re^{-\frac{7}{36}})$ to $O(Re^{-\frac{6}{36}})$, whereas for smaller disturbances virtually no change occurs. Similar features, including the abrupt change, are almost certain to arise in other streaming flows such as the Blasius boundary layer and plane Poiseuille flow as the disturbance size is increased to $O(Re^{-\frac{1}{6}})$. It would be interesting if such an abrupt change in the neutral stability criterion, together with the resulting vast expansion implied in the region of instability in the (α, R)- or (β, R)-plane, is or could be related to the actual behaviour of, say, a spatially growing disturbance as its amplitude passes through the above sizes. A treatment assuming a small departure from the neutral conditions and balances established above and correspondingly slow modulation of the amplitude seems eminently possible then, if instability is encountered close to those conditions, as we would presume. This possibility was anticipated in fact in the second paragraph of § 3, and by the appearance of the perturbation wave speed c_1 in the balance (3.14) that fixes the neutral stability criterion. Again, Benney & Bergeron's (1969) discussions offer some hope for the study of imposed modes more complicated than our single two-dimensional wave. On the other hand, Stewartson's (1978) examination of some not-unrelated critical-layer properties strongly suggests that in the more realistic context of an initial-value problem the temporal development of a disturbance may cause significant departures from the ultimate periodic state assumed here. Finally, the relationship between the present prediction of an order $R^{-\frac{1}{3}}$ disturbance size governing the Rayleigh situation and the classical prediction of $R^{-\frac{2}{3}}$ could prove interesting when other basic flows (e.g. decelerating boundary layers or shear layers) more unstable than those mentioned above are considered.

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REFERENCES

BENNEY, D. J. & BERGERON, R. F. 1969 *Stud. Appl. Math.* **48**, 181.

BODONYI, R. J. & SMITH, F. T. 1981 *Proc. R. Soc. Lond.* A **375**, 65.

BOUTHIER, M. 1973 *J. Méc.* **12**, 75.

BROWN, S. N. & STEWARTSON, K. 1978 *Geophys. Astrophys. Fluid Dyn.* **10**, 1.

BROWN, S. N. & STEWARTSON, K. 1980*a* *J. Fluid Mech.* **100**, 577.

BROWN, S. N. & STEWARTSON, K. 1980*b* *Geophys. Astrophys. Fluid Dyn.* **16**, 171.

DAVIS, R. E. 1969 *J. Fluid Mech.* **36**, 337.

DRAZIN, P. G. & REID, W. H. 1980 *Hydrodynamic Stability.* Cambridge University Press.

FASEL, H., BESTEK, H. & SCHEFENACKER, R. 1977 *AGARD Conf. Proc.* no. 224, paper 14.

GASTER, M. 1974 *J. Fluid Mech.* **66**, 465.

HABERMAN, R. 1972 *Stud. Appl. Math.* **51**, 139.

HABERMAN, R. 1976 *SIAM J. Math. Anal.* **7**.

HALL, P. & SMITH, F. T. 1982 *Stud. Appl. Math.* (to appear).

KELLY, R. E. & MASLOWE, S. A. 1970 *Stud. Appl. Math.* **49**, 301.

LIN, C. C. 1955 *The Theory of Hydrodynamic Stability.* Cambridge University Press.

MASLOWE, S. A. 1972 *Stud. Appl. Math.* **51**, 1.

MASLOWE, S. A. 1977 *Quart. J. R. Met. Soc.* **103**, 769.

MURDOCK, J. W. 1977 *Proc. A.I.A.A. 15th Aerospace Soc. Meeting, A.I.A.A. Paper* no. 77-127.

PEKERIS, C. L. & SHKOLLER, B. 1967 *J. Fluid Mech.* **29**, 31.

REID, W. H. 1965 In *Basic Developments in Fluid Dynamics* (ed. M. Holt), vol. 1. Academic.

REYNOLDS, W. C. & POTTER, M. C. 1967 *J. Fluid Mech.* **27**, 253.

SMITH, F. T. 1979*a* *Proc. R. Soc. Lond.* A **366**, 91.

SMITH, F. T. 1979*b* *Proc R. Soc. Lond.* A **368**, 573; A **371**, 439.

SMITH, F. T. 1980 *Mathematika* **26**, 187; **26**, 211.

SMITH, F. T. & BODONYI, R. J. 1980 *Quart. J. Mech. Appl. Math.* **33**, 293.

SMITH, F. T. & BODONYI, R. J. 1982 To be submitted to *IMA J. Appl. Math.*

STEWARTSON, K. 1978 *Geophys. Astrophys. Fluid Dyn.* **9**, 185.

STUART, J. T. 1963 In *Laminar Boundary Layers* (ed. L. Rosenhead), chap. 9. Oxford University Press.

STUART, J. T. 1960 *J. Fluid Mech.* **9**, 353.

WATSON, J. 1960 *J. Fluid Mech.* **9**, 371.